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Geodesics in the Complex of Curves of a Surface

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Geodesics in the Complex of Curves of a Surface

by

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For my Mom and Dad.

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Geodesics in the Complex of Curves of a Surface

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The University of Texas at Austin, 2002

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The **curve complex** of a closed surface S of genus $g \geq 2$, $\mathcal{C}(S)$, is the complex whose vertices are isotopy classes of simple closed curves on S, and $([x_0], \ldots, [x_n])$ is a simplex of $\mathcal{C}$ if and only if there are disjoint representatives x_i and x_j for all i, j. The curve complex of the torus is similar, with $([x_0], \ldots, [x_n])$ a simplex if and only if there are representatives x_i and x_j which meet in a single point for each $i \neq j$. We use the path metric on $\mathcal{C}(S)$. This dissertation introduces several tools for studying geodesics in the curve complexes of closed orientable surfaces. In the simplest case, when S is a torus, we prove a structure theorem for $\mathcal{C}(S)$ and deduce some results about its global geometry. For the higher genus cases, we introduce two methods for approximating distances. The first yields an elementary proof of the known result [10], [6] that the curve complex has infinite diameter, and a constructive means for estimating distance. The second bounds certain intersection numbers and results in an algorithm to compute distance precisely. All results are expressly constructive and elementary.

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Chapter 1

Introduction

The **curve complex** of a closed surface S of genus $g \geq 2$, $\mathcal{C}(S)$, is the complex whose vertices are isotopy classes of simple closed curves on S, and $([x_0], \ldots, [x_n])$ is a simplex of $\mathcal{C}$ if and only if there are disjoint representatives x_i and x_j for all i, j. The curve complex of the torus is similar, with $([x_0], \ldots, [x_n])$ a simplex if and only if there are representatives x_i and x_j which meet in a single point for each $i \neq j$. We use the path metric on $\mathcal{C}(S)$. This dissertation introduces several tools for studying geodesics in the curve complexes of closed orientable surfaces. In the simplest case, when S is a torus, we prove a structure theorem for $\mathcal{C}(S)$ and deduce some results about its global geometry. For the higher genus cases, we introduce two methods for approximating distances. The first yields an elementary proof of the known result [10], [6] that the curve complex has infinite diameter, and a constructive means for estimating distance. The second bounds certain intersection numbers and results in an algorithm to compute distance precisely. All results are expressly constructive and elementary.

1.1 Historical References

Harvey introduced the curve complex in [4] to study Teichmuller space. Ivanov continued this work in [7], and also studied $Mod(S)$. In [2], Harer proceeds from a cohomological standpoint, and in particular shows the curve complex has the homotopy type of a wedge of spheres of dimension greater than one. Minksy and Masur prove the curve complex is δ-hyperbolic in [10] and provide some further structure theory in [11]. In [9] Feng Luo proves that automorphisms of the curve complex are realized by homeomorphisms of the surface. Hempel studies 3-manifolds using the curve complex in [6].

Our intent is to study the global geometry of the curve complex using only elementary, constructive methods. We begin with a listing of definitions.

1.2 Definitions

We treat graphs somewhat formally. A **graph**, Γ, is a set of **points** (or vertices) with a symmetric **adjacency** relation, usually denoted by $\perp$, with no loops, i.e. for all points x, $x \not\perp x$. Notice, by definition, there are no multiple edges. An **induced subgraph** Γ' of Γ is a graph whose points are a subset of the points of Γ and has adjacency given by the restriction of $\perp$. A **path** of length n in Γ is a sequence $x_0, \ldots, x_n$ so that $x_i \perp x_{i+1}$ for $i = 0, \ldots, n-1$. A graph is **connected** if there is a path between any two points. The **distance** between points $x, y \in \Gamma$, written $d_\Gamma(x, y)$, is the length of a shortest path, or **geodesic**, from x to y. Two graphs are **isomorphic** if there is a bijection on their underlying sets which preserves adjacency.

Naturally associated to each graph is a 1-complex whose path metric restricts to graph distance. Geometric results for graphs tacitly refer to this 1-complex.

A metric space is **geodesic** if the distance between any two points is realized by an isometric embedding of some subinterval of the real line with its standard metric. A 1-complex with the path metric is geodesic. Suppose (X, d) is a geodesic metric space. A **geodesic triangle** in X is a set of three points $a, b, c \in X$ and three **sides**, geodesics g_{ab}, g_{bc}, g_{ac} between their subscripts. For $\delta \geq 0$, a geodesic triangle is **δ-thin** if each point of each side is within δ of some point on one of the other two sides. A geodesic metric space is **δ-hyperbolic** if every geodesic triangle is δ-thin.[1]

A (λ, C) **quasi-isometric embedding** of a metric space (X, d_X) into a metric space (Y, d_Y) is a map $f : X \to Y$ such that for all $x_1, x_2 \in X$

$$\frac{1}{\lambda} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda\, d_X(x_1, x_2) + C.$$

A map, $f : X \to Y$, is **quasi-surjective** if there is a D such that every point of Y is within D of the image of f. Two spaces are **quasi-isometric** if there is a quasi-surjective quasi-isometric embedding of one to the other. A (λ, C) **quasi-geodesic** in X is a (λ, C) quasi-isometric embedding of a (finite or infinite) Euclidean interval into X.

A surface, S, is a compact PL 2-manifold. Isotopy is as in [15], and can always be assumed to be ambient. A simple closed curve is a PL embedded connected 1-manifold. A curve in S is **inessential** if it bounds a disk in S, and **essential** otherwise. All intersections are assumed transverse, also as in [15]. Curves $\alpha, \beta \subseteq S$, meet **inessentially** if there is a disk, called a **bigon**, $B \subseteq S$ with $\partial B = a \cup b$, $a \subseteq \alpha, b \subseteq \beta$. They meet **essentially** if there are no bigons. An **arc** is a PL embedding into S of an interval of the real line. An arc, a, is **inessential** if there is a disk $D \subseteq S$ with $\partial D = a \cup b$ where b is an arc in ∂S, and **essential** otherwise.

The **intersection number** of curves α and β in S, written $i(\alpha, \beta)$, is the minimum number of intersections over all isotopes of α and β. By [5] this number is realized by any pair which meets essentially.

Suppose S is a closed surface of genus g. If $g \geq 2$ then the **curve complex of** S**,** $\mathcal{C}(S)$ is defined as the complex whose vertices are isotopy classes of essential simple closed curves on S. For essential curves $x_i \subseteq S$, $([x_0], \dots, [x_n])$ defines a simplex if $i(x_i, x_j) = 0$ for $i \neq j$. We will use $d(\alpha, \beta)$ to denote the distance between the isotopy classes of α and β in the curve complex with respect to the path metric.

If $g = 1$ then disjoint essential simple closed curves are parallel, so a meaningful definition of the curve complex must be different. In this case, the vertices of $\mathcal{C}(S)$ are isotopy classes of essential simple closed curves, but $([x_0], \dots, [x_n])$ defines a simplex only if $i(x_i, x_j) = 1$ for $i \neq j$. If we use this definition on a higher genus surface, the resulting metric is quasi-isometric to the actual metric in $\mathcal{C}$ for the non-seperating curves, so the accomodation made for the genus 1 case is not drastic.

The $g = 1$ case gets some special verbiage as it is easier to describe. We call the underlying graph of the 1-skeleton of $\mathcal{C}(S)$ the Farey graph in this case.

Since we are concerned with global properties of the metric on the curve complex, it is sufficient to deal with the 1-skeleton. Usually, instead of $\mathcal{C}(S)$ we could write the corresponding graph.

Chapter 2

The Farey Graph

In this chapter we motivate then define a type of graph generalizing trees called *tree-like* graphs. After proving some lemmas about tree-like graphs which help justify their name, we prove that every such graph is hyperbolic. Finally, we prove that the Farey graph is the ascending union of convex tree-like subgraphs and deduce its hyperbolicity, a known result (e.g. [12]).

Isotopy classes of curves on a torus are classified up to sign by the homology classes they represent in the first homology of the torus [14]. If a and b are simple closed curves on a torus which meet in a single point, the classes they represent will generate the first homology of the surface. Any other simple closed curve is then a representative of $p[a] + q[b]$ for some p and q which are relatively prime. We can then identify the set of isotopy classes of curves with the rational numbers, $\mathbb{Q}$, union $\frac{1}{0} = \infty$. We set $\mathcal{F} = \mathbb{Q} \cup \{\infty\}$. For use later, we also define $\mathcal{F}_+ = \mathbb{Q}_{>0}$ and $\mathcal{F}_0 = \mathbb{Q}_{\geq 0} \cup \infty$.

The geometric intersection number on a torus is equal to the absolute value of the algebraic intersection number, so $i(\frac{p}{q}, \frac{r}{s}) = |ps - qr|$.

2.1 $\Gamma(\mathcal{L}, \mathcal{A})$ graphs.

First we prove that $\mathcal{F}_0$ is convex. Suppose $x = \frac{p}{q} \in \mathcal{F}_0$, $y = \frac{r}{s} \in \mathcal{F}$ with $r < 0, s > 0$, and $1 = \Delta(x, y) = |ps - qr| = |ps| + |qr|$. Then one of p or q is zero, and $x = 0$ or ∞. Suppose $x_0, \ldots, x_n$ form a geodesic in $\mathcal{F}$ with $x_0, x_n \in \mathcal{F}_0$. If $x_i \notin \mathcal{F}_0$ for some i then choose $j \leq i \leq k$ so that $x_j, \ldots, x_k \notin \mathcal{F}_0$ and $x_{j-1}, x_{k+1} \in \mathcal{F}_0$. By the previous remark, $x_{j-1}, x_{k+1} \in \{0, \infty\}$, and $x_0, \ldots, x_{j-1}, x_{k+1}, \ldots, x_n$ forms a path *shorter* than our original, contradicting the assumption that it was geodesic. Thus $\mathcal{F}_0$ is convex.

Now we build up to the definition of *tree-like* while proving in the process that $\mathcal{F}_+$ satifies all the requirements. Then, by showing that $\mathcal{F}_0$ can be embedded in $\mathcal{F}_+$, we prove that $\mathcal{F}_0$ is also tree-like.

Let $x = \frac{p}{q} \in \mathcal{F}_+$ where p and q are relatively prime and positive. Now think of performing the Euclidean algorithm on the pair p, q. The first division subtracts a multiple of q from p or vice versa. In other words, we apply one of the following maps to the number x and continue with the resulting numerator and denominator,

$$l^{-1}(x) = x - 1$$

$$r^{-1}(x) = \frac{x}{1 - x}.$$

Since p and q are relatively prime, this process will eventually end with a 1. Inverting, we get a sequence of l's and r's which when applied to 1 gives x, where $l(x) = x + 1$ and $r(x) = \frac{x}{x+1}$.

In fact, there is only *one* such sequence for x. The last map applied in any sequence for x is determined by whether $x = 1, x < 1$, or $x > 1$. If $x = 1$ the sequence is empty. If $x < 1$ the last map applied was r, and if $x > 1$ the last map applied was l. By induction, there is a unique sequence for x which we denote $s(x)$.

For the remainder, we adopt some language from formal language theory. An **alphabet** is a set of symbols, or **letters**, usually finite. A **word** in the alphabet A is a finite sequence of symbols from A. The empty word, or zero length sequence, is denoted by λ. A **language** in the alphabet A is a set of words in the alphabet A. A^* denotes the language of all words in the alphabet A, including λ. If w and w' are words in the alphabet A then ww' denotes the **concatenation** of w and w', the word composed of the letters of w followed by the letters of w'. For all words w, $w\lambda = \lambda w = w$. For $n \in \mathbb{Z}, n > 0$ we let w^n denote the concatenation of w with itself n times, and put $w^0 = \lambda$. Each language has a natural partial order where $w \leq w' \iff$ there is some w'' so that $ww'' = w'$.

Recall the bijection $s : \mathcal{F}_+ \to \{l, r\}^*$ from above for the following.

Lemma 2.1.1. *Let $x, y \in \mathcal{F}_+$. Then $x \perp y$ if and only if $s(x) = s(y)lr^k, s(x) = s(y)rl^k, s(y) = s(x)lr^k$, or $s(y) = s(x)rl^k$ for some $k \geq 0$.*

Proof. Suppose $x, y \in \mathcal{F}_+$ and $x \perp y$. If $s(x)$ and $s(y)$ begin with different letters, then we have some $p, q, r, s > 0$ such that

$$\begin{aligned} 1 &= \Delta\left(l\left(\frac{p}{q}\right), r\left(\frac{s}{t}\right)\right) = \Delta\left(\frac{p+q}{q}, \frac{s}{s+t}\right) \\ &= |(p+q)(s+t) - qs| = ps + pt + qt \geq 3 \end{aligned}$$

which can not happen. So either $s(x)$ and $s(y)$ begin with the same letter, in which case we cancel and induct, or one of $s(x)$ or $s(y)$ is empty. It is easy to compute that the only numbers adjacent to $1 = s^{-1}(\lambda)$ are $\frac{k+2}{k+1} = s^{-1}(lr^k)$ and $\frac{k+1}{k+2} = s^{-1}(rl^k)$. The result follows. $\square$

In abstract terms, we have the following.

Definition 2.1.2. *Let $\mathfrak{L}$ and $\mathcal{A}$ be languages on the same alphabet.* $\Gamma(\mathfrak{L}, \mathcal{A})$ *is defined to be the graph with vertex set $\mathfrak{L}$ where $w, w' \in \mathfrak{L}$ are adjacent iff $w = w'x$ or $w' = wx$ for some $x \in \mathcal{A}$.*

In a trivial way, every graph with countably many vertices is actually a $\Gamma(\mathfrak{L}, \mathcal{A})$ graph. Suppose Γ is a graph with vertex set $v(\Gamma) = \{x_0, x_1, \ldots\}$. Define the alphabet $A = v(\Gamma)$. Put $\mathfrak{L} = \{x_0, x_0x_1, x_0x_1x_2, \ldots\}$ and $\mathcal{A} = \{x_{i+1} \cdots x_j | \text{ where } i < j \text{ and } x_i \text{ is adjacent to } x_j\}$. Then $\Gamma \cong \Gamma(\mathfrak{L}, \mathcal{A})$.

Furthermore, put $B = \{b, 0, 1\}$. Define $f : A \to B^*$ so that $f(x_i)$ is the letter "b" followed by the binary representation of i. Now extend f to A^* by concatenation. Put $\mathfrak{L}' = f(\mathfrak{L}), \mathcal{A}' = f(\mathcal{A})$. It can be shown that $\Gamma \cong \Gamma(\mathfrak{L}', \mathcal{A}')$, where $\mathfrak{L}'$ and $\mathcal{A}'$ are languages on a *finite* alphabet.

So a $\Gamma(\mathfrak{L}, \mathcal{A})$ structure is not restrictive. We will proceed, therefore, by analogy with a natural representation of trees as $\Gamma(\mathfrak{L}, \mathcal{A})$ graphs.

By a **simple** $\Gamma(\mathcal{L}, \mathcal{A})$ graph we mean one with $\mathcal{L} = A^*$ and $\mathcal{A} = A$ for a finite alphabet A. So any tree is an induced subgraph of a simple $\Gamma(\mathcal{L}, \mathcal{A})$ graph. It is easy enough to prove in general that a tree is hyperbolic and has a *unique* geodesic between any two points, but by proving the same results for simple $\Gamma(\mathcal{L}, \mathcal{A})$ graphs, a methodology emerges which can be adapted for the larger class of *tree-like* graphs.

Definition 2.1.3. *We say that a* $\Gamma(\mathcal{L}, \mathcal{A})$ *graph is* **tree-like** *if* $\mathcal{A}$ *is closed under right cancellation, i.e. if* $y, xy \in \mathcal{A}$ *and* $x \neq \lambda$ *then* $x \in \mathcal{A}$*. Graphs isomorphic to tree-like* $\Gamma(\mathcal{L}, \mathcal{A})$ *graphs are also called tree-like.*

We have shown that $\mathcal{F}_+ \cong \Gamma\left(\{l, r\}^*, \{lr^k, rl^k |\ k \geq 0\}\right)$ with s the isomorphism. So $\mathcal{F}_+$ is clearly tree-like.

Now we want a tree-like $\Gamma(\mathcal{L}, \mathcal{A})$ structure on $\mathcal{F}_0$, our convex piece of $\mathcal{F}$. To do this, we map $\mathcal{F}_0$ injectively into $\mathcal{F}_+$ and use the structure from the image. The map we use is $l \circ r$. Recall the maps l and r from earlier, so that $l \circ r(x) = \frac{2x+1}{x+1}$. Let $p, q, r, s \in \mathbb{Z}$.

$$\begin{aligned}
\Delta\left(l \circ r\left(\frac{p}{q}\right), l \circ r\left(\frac{r}{s}\right)\right) &= \Delta\left(\frac{2p+q}{p+q}, \frac{2r+s}{r+s}\right) \\
&= |(r+s)(2p+q) - (2r+s)(p+q)| \\
&= |ps - rq| \\
&= \Delta\left(\frac{p}{q}, \frac{r}{s}\right)
\end{aligned}$$

So $l \circ r$ is an autmorphism of $\mathcal{F}$. In particular, $s \circ l \circ r(\mathcal{F}_+) = \{lrw |\ w \in \{l, r\}^*\}$ since $s(l \circ r(x)) = lr\, s(x)$. Also, $s \circ l \circ r(0) = s(1) = \lambda$ and $s \circ l \circ r(1) = s(2) = l$.

So $\mathcal{F}_0 \cong \Gamma\left(\{\lambda, l\} \cup \{lrw|\ w \in \{l,r\}^*\}, \{lr^k, rl^k|\ k \geq 0\}\right)$ with $s \circ l \circ r$ the isomorphism. Clearly $\mathcal{F}_0$ is tree-like.

2.2 Tree-like graphs.

The lemmas which follow are generalizations of facts about trees when viewed as simple $\Gamma(\mathcal{L}, \mathcal{A})$ graphs. Later in this section we classify the geodesics of tree-like graphs and prove that all tree-like graphs are $\frac{3}{2}$-hyperbolic. For the remainder, fix $\mathcal{L}$ and $\mathcal{A}$ so that $\Gamma \cong \Gamma(\mathcal{L}, \mathcal{A})$ is tree-like and connected.

Definition 2.2.1. *For $w, w' \in \mathcal{L}$, write $w \swarrow w'$ (also $w' \searrow w$) if $w < w'$ and $w \perp w'$. Write $w \underset{=}{\swarrow} w'$ (also $w' \underset{=}{\searrow} w$) if $w = w'$ or $w \swarrow w'$.*

If $\{w'|\ w' \swarrow w\} \neq \emptyset$, then define $c(w) = \max\{w'|\ w' \swarrow w\}$ and $C(w) = \min\{w'|\ w' \swarrow w\}$. Finally, define $\mathcal{D}(w) = \{w, c(w), c^{(2)}(w), \ldots\}$.

Lemma 2.2.2. ("the apex lemma") $x, y \underset{=}{\swarrow} z \implies x = y$ *or* $x \perp y$.

Proof. If $x = z$ or $y = z$ the result is obvious. So $z = xw = yw'$ for some words $w, w' \in \mathcal{A}$. If $w = w'$ then $x = y$. If w' is longer than w, say, we can write $w' = w''w$. Then $x = yw''$, but since Γ is tree-like $w'' \in \mathcal{A}$, and $x \perp y$. $\square$

Lemma 2.2.3. ("common descent") *Suppose $x_n \searrow \cdots \searrow x_0$. Then for $i = 0, \ldots, n$, $x_i \in \mathcal{D}(x_n)$. Also, for each $c \in \mathcal{D}(x_n)$ with $c \geq x_0$ either $c = x_n$ or there is some i so that $x_{i+1} > c \underset{=}{\searrow} x_i$.*

Proof. We induct on $|\{c \in \mathcal{D}(x_n)|\ c \geq x_0\}|$. Put $c_0 = c(x_n)$. By definition of c_0 either $c_0 = x_{n-1}$ or $c_0 > x_{n-1}$. If $c_0 = x_{n-1}$ then remove x_n and invoke the inductive hypothesis with $x_{n-1} \searrow \cdots \searrow x_0$ recalling $\mathcal{D}(x_n) = \{x_n\} \cup \mathcal{D}(c(x_n))$ (see Figure 2.1(a)). If $c_0 > x_{n-1}$ then the apex lemma applied to $c_0, x_{n-1} \overset{\swarrow}{=} x_n$ implies $c_0 \searrow x_{n-1}$. So we can apply the inductive hypthesis to $c_0 \searrow x_{n-1} \searrow \cdots \searrow x_0$ (see Figure 2.1(b)). The base case is obvious. $\square$

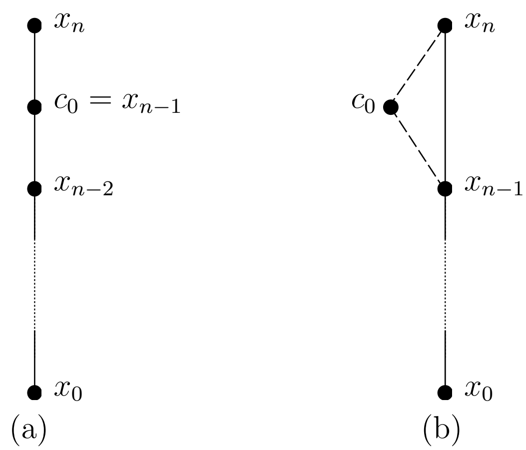

Figure 2.1: The cases for common descent.

In a simple $\Gamma(\mathcal{L}, \mathcal{A})$ graph, each vertex has a unique "root path", the path from the vertex to the empty word. To compare two vertices, we compare their root paths. $\mathcal{D}$ of a vertex is the analog in a tree-like graph of a root path. The comparison in both cases is the *meet*. If $(\mathcal{D}(x) \cap \mathcal{D}(y)) \neq \emptyset$ we define the **meet** of x and y, $x \wedge y = \max(\mathcal{D}(x) \cap \mathcal{D}(y))$. In the following theorem, we see that the meet always exists in connected tree-like graphs.

Theorem 2.2.4. *A geodesic in Γ between vertices a and b takes one of three forms, up to exchanging a with b:*

- [monotonic] $a = x_0 \searrow \cdots \searrow x_n = b$ *where* $a \wedge b = b$,

- ["V"-shaped (a)] $a = x_0 \searrow \cdots \searrow x_i \swarrow \cdots \swarrow x_n$ *where* $x_i \underset{=}{\swarrow} (a \wedge b)$ *and* $x_j > (a \wedge b)$ *for* $j \neq i$,

- ["V"-shaped (b)] $a = x_0 \searrow \cdots \searrow x_{i-1} \searrow x_i \swarrow \cdots \swarrow x_n$ *where* $x_{i-1}, x_i \underset{=}{\swarrow} (a \wedge b)$ *and* $x_j > (a \wedge b)$ *for* $j \neq i-1, i$.

See Figure 2.2.

Proof. Let $a = x_0, \ldots, x_n = b$ be (the vertices of) a geodesic. By the apex lemma, any triple along a geodesic is either monotonic or "V" shaped. Allowing for $i = 0$ or n, we have $a = x_0 \searrow \cdots \searrow x_i \swarrow \cdots \swarrow x_n = b$. By common descent, for $j = 0 \ldots i$, $x_j \in \mathcal{D}(a)$ and for $j = i \ldots n$, $x_j \in \mathcal{D}(b)$. So $x_i \in (\mathcal{D}(a) \cap \mathcal{D}(b))$ and $x_i \leq (a \wedge b)$. By common descent, there is $i_0 \leq i$ so that $(a \wedge b) \underset{=}{\searrow} x_{i_0}$ and either $i = 0$ or $x_{i_0 - 1} > (a \wedge b)$. Similarly, there is $i_1 \geq i$ so that $x_{i_1} \underset{=}{\swarrow} (a \wedge b)$ and either $i_1 = n$ or $(a \wedge b) < x_{i_1+1}$. By the apex lemma, therefore, either $x_{i_0} = x_{i_1}$ or $x_{i_0} \perp x_{i_1}$. The cases are enumerated as in the statement of the theorem. $\square$

Remark. If γ is a geodesic between a and b, we call the longest descending subpath of γ containing a the "a half" of γ. Similarly for b. By the previous theorem, the two halves intersect in one point.

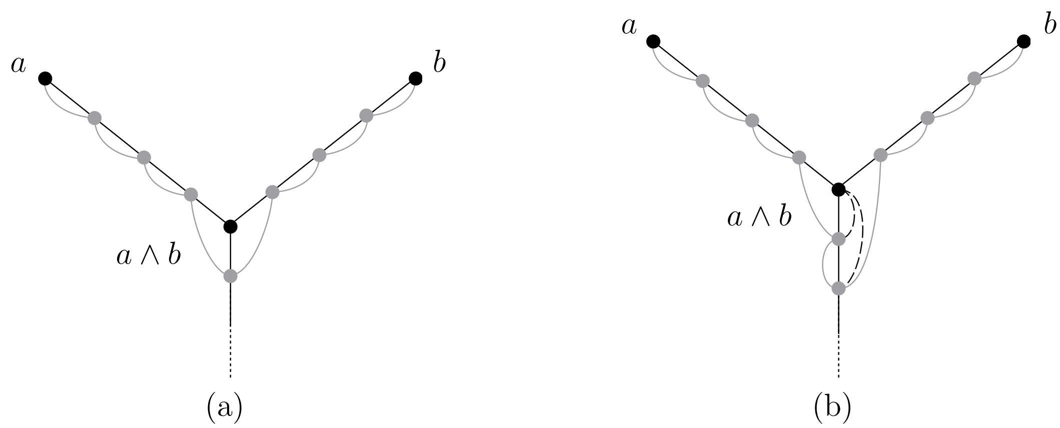

Figure 2.2: The non-monotonic geodesics.

Notice that the points a and b do not necessarily determine the type of geodesic between a and b. There are both a monotonic and a "V"-shaped (b) type geodesic between "lrr" and "l" in $\mathfrak{F}_0$. There is, however, a vertex-by-vertex "standardization" of an arbitrary geodesic as in the following corollary.

Corollary 2.2.5. *Let $a, b \in \mathfrak{L}$. Choose m minimal such that $C^{(m+1)}(a) \leq (a \wedge b)$ (or zero if $a = (a \wedge b)$) and n minimal such that $C^{(n+1)}(b) \leq (a \wedge b)$ (or zero if $b = (a \wedge b)$). Then, up to exchanging a and b exactly one of the following is a geodesic:*

- $a \searrow C(a) \searrow \cdots \searrow C^{(m)}(a) = b,$
- $a \searrow C(a) \searrow \cdots \searrow C^{(m)}(a) \searrow x \swarrow C^{(n)}(b) \swarrow \cdots \swarrow b,$
- $a \searrow C(a) \searrow \cdots \searrow C^{(m)}(a) \searrow x \searrow y \swarrow C^{(n)}(b) \swarrow \cdots \swarrow b,$

for some $x, y \overset{\swarrow}{\leqq} (a \wedge b)$.

Proof. Suppose γ is a geodesic between points a and b, where the vertices of γ are $a = x_0, \ldots, x_n = b$. If $C(x_0) > (a \wedge b)$ then $x_1 \overset{\searrow}{=} C(x_0)$ by the apex lemma applied to $x_1, C(x_0) \overset{\swarrow}{=} x_0$ and definition of $C(x_0)$. Since $x_1 > (a \wedge b)$ the next vertex, x_2 must be smaller than x_1. Apply the apex lemma to $C(x_0), x_2 \overset{\swarrow}{=} x_1$ to get that $C(x_0) \perp x_2$ and we can replace x_1 in γ with $C(x_0)$. Continuing this way, we can replace all of the vertices on the a half of γ above $a \wedge b$ with iterates of C applied to a. Repeat for the b half of γ. By the previous theorem, at most two vertices of γ remain, which we name x and y. It is plain to see that the three types are mutually exclusive. $\square$

Proposition 2.2.6. *(e.g. [12])* Γ *is* $\frac{3}{2}$*-hyperbolic.*

Proof. Let $a, b, c \in \mathcal{L}$. Suppose $\gamma_{ab}, \gamma_{ac}, \gamma_{bc}$ are geodesics forming a triangle with subscripts denoting endpoints. Our goal is to show that each geodesic is contained in a $\frac{3}{2}$ neighborhood of the union of the other two. We will show that each vertex in the a half of γ_{ab} is adjacent to a vertex of either γ_{ac} or γ_{bc}. The general result follows.

Case I: Assume $(a \wedge b) \geq (a \wedge c)$. Then the nadir of γ_{ac} is less than or equal to $(a \wedge b)$. By common descent, for every vertex $x \in \mathcal{D}(a)$ such that $x \geq (a \wedge b)$ there is a vertex y_x in the a half of γ_{ac} so that $y_x \overset{\swarrow}{=} x$. Theorem 2.2.4 implies that for each vertex x in the a half of γ_{ab} either $x \geq (a \wedge b)$ or $x \swarrow (a \wedge b)$. For the first class of vertices we have $y_x \overset{\swarrow}{=} x$. For the second class, we apply the apex lemma to $x, y_{(a\wedge b)} \overset{\swarrow}{=} (a \wedge b)$ to get $x = y_{(a\wedge b)}$ or $x \perp y_{(a\wedge b)}$.

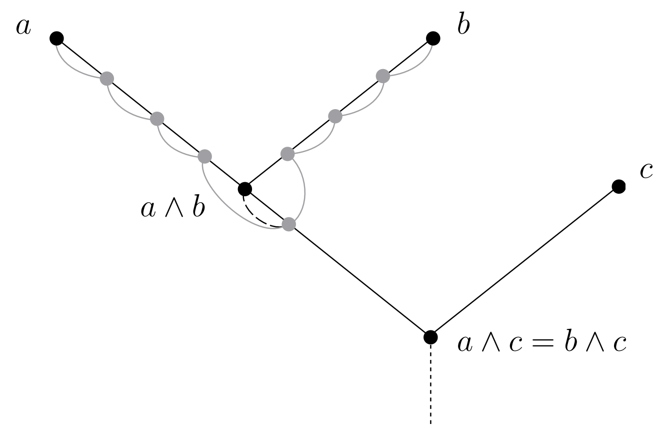

Figure 2.3: Case I: $(a \wedge b) \geq (a \wedge c)$

Case II: Assume $(a \wedge b) < (a \wedge c)$. We distinguish two classes of vertices in the a half of γ_{ab}: $x \geq (a \wedge c)$ and $x < (a \wedge c)$. The first class is handled with γ_{ac} in the same way as Case I.

Let x_0 be the largest second class vertex. Common descent implies that $x_0 \swarrow (a \wedge c)$. Now construct γ'_{ab} by replacing the first class vertices in the a half of γ_{ab} with $(a \wedge c)$. So γ'_{ac} is a path descending from $(a \wedge c)$ which contains all vertices from the second class. Similarly construct γ'_{bc} from the c half of γ_{bc} by replacing all vertices greater than $a \wedge c$ with $a \wedge c$. Then since $(b \wedge c) = (a \wedge b)$, we can proceed with γ'_{ab} and γ'_{bc} as in Case I to show that the second class vertices are adjacent to vertices of γ_{bc}. $\square$

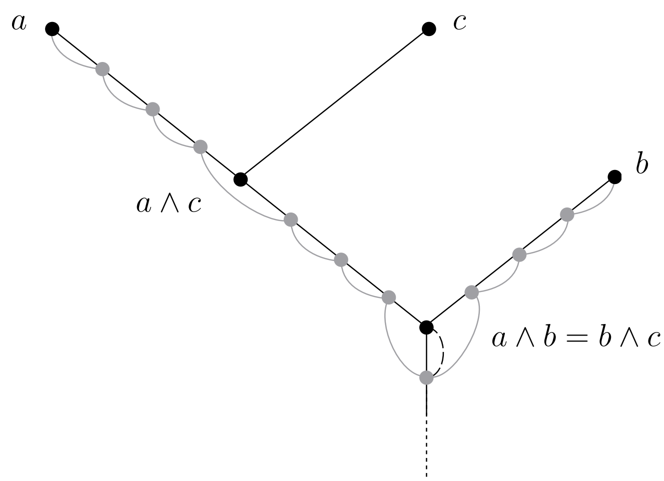

Figure 2.4: Case II: $(a \wedge b) < (a \wedge c)$

Now we mention a way of generalizing this result somewhat. If $\mathcal{A}$ is closed under right cancellation then $\Gamma(\mathfrak{L}, \mathcal{A})$ will be tree-like. If this condition fails, but in a "bounded" way, we can still reclaim some global geometry.

Proposition 2.2.7. *Let $\mathfrak{L}$, $\mathcal{A}$, and $\mathcal{A}'$ be languages on the same alphabet such that $\mathcal{A}' \supseteq \mathcal{A}$, and for all $x \in \mathfrak{L}, w \in \mathcal{A}'$, we have $xw \in \mathfrak{L}$. Also assume that each point of $\mathcal{A}'$ is within N of λ in $\Gamma(\mathcal{A}', \mathcal{A})$. Then $\Gamma(\mathfrak{L}, \mathcal{A})$ and $\Gamma(\mathfrak{L}, \mathcal{A}')$ are $(N, \frac{1}{2})$ quasi-isometric.*

Proof. Let d_Γ denote the distance function in Γ. Clearly $d_{\Gamma(\mathfrak{L},\mathcal{A}')} \leq d_{\Gamma(\mathfrak{L},\mathcal{A})}$ since vertices adjacent in $\Gamma(\mathfrak{L}, \mathcal{A})$ are also adjacent in $\Gamma(\mathfrak{L}, \mathcal{A}')$. Now suppose $x \perp y$ in $\Gamma(\mathfrak{L}, \mathcal{A}')$. Without loss of generality, assume $y = xw$ for some $x \in \mathcal{A}'$. By the hypothesis, there is a sequence $\lambda = w_0 \perp \cdots \perp w_n = w$ in $\Gamma(\mathcal{A}', \mathcal{A})$ where $n < N$. Then $x = xw_0 \perp \cdots \perp xw_n = xw = y$ is a path in

$\Gamma(\mathcal{L},\mathcal{A})$, and $d_{\Gamma(\mathcal{L},\mathcal{A})}(x,y) \leq N$. By extension, therefore, for any $x, y \in \mathcal{L}$ we have $\frac{1}{N} d_{\Gamma(\mathcal{L},\mathcal{A})}(x,y) \leq d_{\Gamma(\mathcal{L},\mathcal{A}')}(x,y)$. To account for edges of $\Gamma(\mathcal{L},\mathcal{A}')$ not in $\Gamma(\mathcal{L},\mathcal{A})$ we add $\frac{1}{2}$. $\square$

So if, for example, $\mathcal{A}'$ is closed under right cancellation, then $\Gamma(\mathcal{L},\mathcal{A}')$ is tree-like, and therefore hyperbolic. The proposition then implies that $\Gamma(\mathcal{L},\mathcal{A})$ is hyperbolic with a proportional constant.

2.3 Putting $\mathcal{F}$ back together.

Now, we piece together $\mathcal{F}$ using copies of $\mathcal{F}_0$. Basically, $\mathcal{F}$ is an ascending union of convex subgraphs each isomorphic to $\mathcal{F}_0$. The intent of the following, however, is to provide an abstract context where $\mathcal{F}$ can be contructed without knowing, a priori, that $\mathcal{F}_0 \subseteq \mathcal{F}$. In keeping the presentation self-contained, the possibly more familiar language of category theory is replaced by more natural terminology (no pun intended).

All of the *maps* in the following are assumed to preserve adjacency. That is, if $f : \Gamma \to \Gamma'$ is a map and x and y are adjacent in Γ then $f(x)$ and $f(y)$ are adjacent or equal in Γ'.

Let $\{\Gamma_i | \, i \in I\}$ be a collection of graphs. Suppose $\{f_{ij} : \Gamma_i \to \Gamma_j | \, (i,j) \in F \subseteq I \times I\}$ is a set of maps such that $(i,j),(j,k) \in F \implies (i,k) \in F$ and $f_{jk} \circ f_{ij} = f_{ik}$. Then we call $(\{\Gamma_i\}, \{f_{ij}\})$ a **diagram**. We assume, without loss of generality, that $(i,i) \in F$ and $f_{ii} = \mathrm{Id}_{\Gamma_i}$ for all i.

Given a diagram, $(\{\Gamma_i\}, \{f_{ij}\})$, suppose there is a graph Γ and maps

$\{g_i : \Gamma_i \to \Gamma\}$ such that $g_j \circ f_{ij} = g_i$ for all i, j. Then we say $(\Gamma, \{g_i\})$ is a **weak union** of the diagram $(\{\Gamma_i\}, \{f_{ij}\})$.

Suppose $(\Gamma, \{g_i\})$ is a weak union such that for any weak union, $(\Gamma', \{g_i'\})$, there exists a *unique* map $p : \Gamma \to \Gamma'$ so that $g_i' = p \circ g_i$ for all i. We call $(\Gamma, \{g_i\})$ a **union** of $(\{\Gamma_i\}, \{f_{ij}\})$.

Proposition 2.3.1. *Every diagram has a unique union.*

Proof. If there is a union, it is not hard to show that it is unique directly from the definition. If Γ and Γ' are unions, there are maps $p : \Gamma \to \Gamma'$ and $p' : \Gamma' \to \Gamma$ as in the definition. Uniqueness implies that $p \circ p' = \mathrm{Id}_\Gamma$ and $p' \circ p = \mathrm{Id}_{\Gamma'}$, so Γ and Γ' are isomorphic.

We define an equivalence relation on $\coprod_{i=0}^{\infty} \Gamma_i$, the disjoint union of the Γ_i. For $x \in \Gamma_i$ and $y \in \Gamma_j$, we write $x \sim_0 y$ if and only if $f_{ij}(x) = y$. Let $\sim$ be the symmetric, transitive closure of $\sim_0$. Define Γ to be the set of equivalence classes of $\sim$ and define $g_i : \Gamma_i \to \Gamma$ by $g_i(x) = [x]$.

By definition, $(\Gamma, \{g_i\})$ is clearly a weak union. Let $(\Gamma', \{g_i'\})$ be another weak union. To satisfy the definition of union above, the map p must be defined so that for $x \in \Gamma_i$, $p([x]) = p(g_i(x)) = g_i'(x)$. If we show that p is well defined, we are done. This follows from the fact that $(\Gamma', \{g_i'\})$ is a weak union, i.e. $g_j'(f_{ij}(x)) = g_i'(x)$. □

A diagram, $(\{\Gamma_i\}, \{f_{ij}\})$, is **geometric** if each f_{ij} is injective with a convex image (in Γ_j) and for each pair i, j there is some k so that $f_{ik} : \Gamma_i \to \Gamma_k$ and $f_{jk} : \Gamma_j \to \Gamma_k$ are maps of the diagram.

Proposition 2.3.2. *Suppose* $(\{\Gamma_i\}, \{f_{ij}\})$ *is a geometric diagram and* $(\Gamma, \{g_i\})$ *is its union. Then for each* i*,* g_i *is an isomorphism onto its image, its image is convex in* Γ*, and* $\Gamma = \bigcup_{i\in I} g_i(\Gamma_i)$*.*

In particular, each (finite) geodesic in Γ *is the image of a geodesic in some* Γ_i*. Conversely, the image of each geodesic is a geodesic. Furthermore, if each* Γ_i *is* δ*-hyperbolic, then so is* Γ*.*

Proof. First we show that $g_{i_0}(x) = g_{i_0}(y) \implies x = y$, i.e. $g_{i_0} : \Gamma_{i_0} \to \Gamma$ is injective. Suppose $x, y \in \Gamma_{i_0}$ and $x \sim y$. Without working much beyond the definition of $\sim$, we can show there is a set of points $x_j \in \Gamma_{i_j},\ j = 0 \ldots n$ such that $x_0 = x, x_n = y$ and for $j \leq (n-2)$ we have

$$x_j \sim f_{i_j i_{j+1}}(x_j) = x_{j+1} = f_{i_{j+2} i_{j+1}}(x_{j+2}) \sim x_{j+2}.$$

Using geometricity, we can (inductively) find a k so that there are maps $f_{i_j k}$ in the diagram for all j. We can then show, inducting again, that $f_{i_0 k}(x_0) = f_{i_j k}(x_j)$ for all $j = 0 \ldots n$. The result that $f_{i_0 k}(x) = f_{i_0 k}(x_0) = f_{i_n k}(x_n) = f_{i_0 k}(y)$ along with the injectivity of $f_{i_0 k}$ then imply $x = y$.

Similarly, if $x, y \in \Gamma_{i_0}$ and $[x] = [x_0], \ldots, [x_n] = [y]$ with $x_j \in \Gamma_{i_j}$ forms a geodesic in Γ we can find a k so that $f_{i_j k}$ are maps in our diagram for all j. Then since the image of $f_{i_0 k}$ is convex containing x and y, our geodesic must be the image under $f_{i_0 k}$ of a geodesic of Γ_{i_0}.

Again, using a triangle instead of a geodesic, we can show that each triangle in Γ is the image of a triangle in Γ_i for some i. $\square$

We show now that the Farey graph is abstractly a union of copies of $\mathcal{F}_0$. Put $\Gamma_0 = \Gamma\left(\{\lambda, l\} \cup \{lrw |\ w \in \{l,r\}^*\}, \{lr^k, rl^k |\ k \geq 0\}\right)$. Earlier we showed that $\mathcal{F}_0 \cong \Gamma_0$ via the isomorphism $s \circ l \circ r : \mathcal{F}_0 \to \Gamma_0$, which we denote t for convenience.

Let $\{\Gamma_i |\ i \in \mathbb{Z}_{\geq 0}\}$ be a collection of pairwise disjoint graphs each isomorphic to Γ_0. We will suppress the isomorphisms $\Gamma_i \to \Gamma_0$, since intent should be clear from context. Define $f_{i(i+n)} : \Gamma_i \to \Gamma_{i+n}$ by $f_{i(i+n)}(w) = (lr)^n w$ for $n \geq 0$. From earlier results, it follows that $(\{\Gamma_i\}, \{f_{i(i+n)}\})$ is a geometric diagram.

Define $g_i : \Gamma_i \to \mathcal{F}$ by $g_i(w) = (l \circ r)^{-i} \circ t^{-1}(w)$. Then

Proposition 2.3.3. *$(\mathcal{F}, \{g_i\})$ is the union of $(\{\Gamma_i\}, \{f_{i(i+n)}\})$.*

Proof. Notice that t^{-1} "commutes" with $l \circ r$ in the following sense, $(l \circ r) \circ t^{-1}(w) = t^{-1}(lrw)$ for any $w \in \Gamma_0$. We compute,

$$\begin{aligned} g_i(w) &= (l \circ r)^{-i} \circ t^{-1}(w) = (l \circ r)^{-(i+n)} \circ (l \circ r)^n \circ t^{-1}(w) \\ &= (l \circ r)^{-(i+n)} \circ t^{-1}((lr)^n w) = g_{i+n} \circ f_{i(i+n)}(w), \end{aligned}$$

so $(\mathcal{F}, \{g_i\})$ is a weak union.

To show that it is the actual union, we first show that each point in $\mathcal{F}$ can be written as $(l \circ r)^{-i} \circ t^{-1}(w)$ for some $i \geq 0$ and $w \in \Gamma_0$.

Let $x \in \mathcal{F}$. If $x \in \mathcal{F}_0$ then $x = t^{-1}(w)$ where $w = t(x) \in \Gamma_0$ is defined. So we can assume $x = \frac{-p}{q}$ for relatively prime integers $p, q > 0$. Then $(l \circ r)(\frac{p}{q}) = \frac{2p-q}{p-q}$ is also in lowest terms. If $p \geq q$ or $p < q$ and $2p \leq q$ then

$(l\circ r)(\frac{p}{q}) \in \mathcal{F}_0$. So we can assume $p < q$ and $2p > q$. Then $|p-q| = q-p < q = |q|$ and $|2p-q| = 2p-q = p-(q-p) < p = |-p|$. In other words, applying $(l\circ r)$ decreases both the absolute value of the numerator and the absolute value of the denominator in lowest terms, and we can induct.

For each $x \in \mathcal{F}$ we have found some $i \geq 0$ so that $(l\circ r)^i(x) \in F_0$, and we can find $w \in \Gamma_0$ so that $t^{-1}(w) = (l\circ r)^i(x)$. Equivalently $x = (l\circ r)^{-i}\circ t^{-1}(w)$, the desired result.

Now let $(\Gamma', \{g_i'\})$ be an arbitrary weak union of $(\{\Gamma_i\}, \{f_{i(i+n)}\})$. We need to find $p : \mathcal{F} \to \Gamma'$ so that $g_i'(w) = p\circ g_i(w) = p((l\circ r)^{-i}t^{-1}(w))$. In fact, this necessary condition suffices for a definition if we can show that it is well-defined. Suppose $(l\circ r)^{-i}t^{-1}(w) = (l\circ r)^{-i'}t^{-1}(w')$ for some $i, i' \geq 0$ and $w, w' \in \mathcal{F}_0$. Assume without loss of generality that $i' \geq i$, then

$$\begin{aligned} w' &= t\left((l\circ r)^{i'-i}t^{-1}(w)\right) = (lr)^{i'-i}w \text{ or equivalently,} \\ w' &= f_{i(i+(i'-i))}(w). \end{aligned}$$

So if $(l\circ r)^{-i}t^{-1}(w) = (l\circ r)^{-i'}t^{-1}(w')$ then $g_{i'}'(w') = g_{i'}'\circ f_{ii'}(w) = g_i'(w)$ since $(\Gamma', \{g_i'\})$ is a weak union, completing the proof. $\square$

2.4 Conclusion

Notice that the maps l and r are actually Dehn twists along the so-called "basis" curves, 0 and ∞. Then $\mathcal{F}_0$, the polyhedron of a (degenerate) train track, is "generated" by applications of l and r to 1. The whole of $\mathcal{F}$ is reclaimed by "pushing" any curve into $\mathcal{F}_0$ using $l \circ r$.[13, 14]

In the case of the curve complex of a higher genus surface, we might look to do something similar. Find a "basis" of curves to twist along which can describe the curves of some convex piece of the complex, possibly a set of "independent" curves carried by a large train track. Then by applying a set of pseudo-Anosov maps with stable laminations carried by the same track, for example, "push" all other curves into our convex piece.[8]

Chapter 3

Curve complexes of surfaces of genus ≥ 2.

This chapter is significantly more geometric. We work in the piecewise-linear category (see [15], for example), although the cutting and pasting operations employed are easily adapted to other categories. Throughout the chapter S will denote a closed, oriented surface of genus ≥ 2, and curves are assumed to be essential and in general position where applicable.

In Lemma 2.1 of [6], Hempel produces a path in the curve complex between any two vertices, whose length is bounded by a function of their intersection number:

Lemma 3.0.1. *For $\alpha, \beta \in \mathcal{C}(S)$ we have*

$$\begin{aligned} d(\alpha, \beta) &\leq 2 + 2\log_2(i(\alpha, \beta)) \\ \text{(equivalently } 2^{d(\alpha,\beta)} &\leq 4i(\alpha, \beta)^2) \end{aligned}$$

The constructed path, $\alpha = x_0, \ldots, x_n = \beta$, is such that for all k, $i(\alpha, x_k), i(x_k, \beta) \leq i(\alpha, \beta)$ and as a function of k, $i(x_k, \beta)$ is monotonically decreasing. We do not, however, get any idea of how good an approximation n is of $d(\alpha, \beta)$. This chapter attempts to construct paths with length approximating $d(\alpha, \beta)$ for which each of the inequalities above holds.

First an example. For any N we can construct a pair of curves, α and β, such that $d(\alpha, \beta) \leq 4$ and any curve disjoint from α must intersect β at least N times.

Let N be some large integer. Figure 3.1 describes a genus two surface, call it S, where the numbers denote the attaching spheres of the 1-handles. The curves a, b, c, d are labeled.

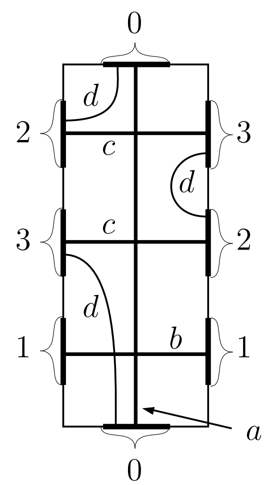

Figure 3.1: Curves for example.

Set $\alpha = a$ and $\beta = \tau_c^{(\frac{N}{2})}\tau_b^{(N)}(a)$. We claim $d(\alpha, \beta) \leq 4$. Clearly, from Figure 3.1, $d(a, d) = d(d, c) = d(b, c) = 1$ and $d(a, c) = d(a, b) = 2$.

$$\begin{aligned} d(a, \tau_c^{(\frac{N}{2})}\tau_b^{(N)}(a)) &\leq d(a, b) + d(b, \tau_c^{(\frac{N}{2})}\tau_b^{(N)}(a)) && (3.1) \\ &\leq 2 + d(b, \tau_c^{(\frac{N}{2})}\tau_b^{(N)}(a)) && (3.2) \\ &= 2 + d(b, a) \leq 2 + 2 = 4 && (3.3) \end{aligned}$$

Equation (3.1) is the triangle inequality, (3.2) substitution. Since $i(c, b) = i(b, b) = 0$, the homeomorphism $(\tau_c^{(\frac{N}{2})}\tau_b^{(N)})^{-1}$ fixes b, and (3.3) follows.

On the other hand, any curve γ disjoint from α must have $i(\beta, \gamma) \geq N$. If we cut S along α we get a torus with two boundary components, call it S'. The arcs of β/α fall into three families of at least N parallel arcs each, corresponding to the 1-handles in the diagram. Since these arcs fill S' and γ is essential, γ must meet at least one family, and therefore β, in at least N points.

These examples illustrate that intersection numbers between points along a geodesic with its endpoints can be arbitrarily large, irrespective of distance. An examination of the way that intersections are forced in these examples, however, leads to the technique of the next section. There we will give an elementary proof of the known result [10],[6] that the curve complex has infinite diameter, and a construction addressing monotonicity of intersection numbers along a path (see Proposition 3.1.5).

The second section gives an unrelated method which is used to produce an algorithm for determining the exact distance. The method is also used to prove the existence of a geodesic satisfying inequalities similar to those in the lemma above (see Corollary 3.2.4).

The results of this chapter were intended to provide insight into a combinatorial proof of hyperbolicity of $\mathcal{C}(S)$, therefore both sections stress the methods employed over their example applications. Some questions for continuing investigation are asked in the conclusion of the chapter.

3.1 First method

First we give a combinatorial proof that the curve complex has infinite diameter. The construction depends on an explicit version of the Dehn twist, defined next.

Definition 3.1.1. *A* **standard Dehn twist** *of a curve β about a curve α is obtained as follows, assuming α and β meet essentially. Take β' to be a parallel copy of β, and A a closed annular neighborhood of α meeting β and β' in spanning arcs. Let $\alpha_1 \ldots \alpha_n$ be n parallel copies of α in A, where $n = i(\alpha, \beta)$. The twist is obtained from $\alpha_1 \cup \cdots \cup \alpha_n \cup \beta'$ by resolving each intersection as in Figure 3.2. The result is denoted by $\tau_\alpha(\beta)$.*

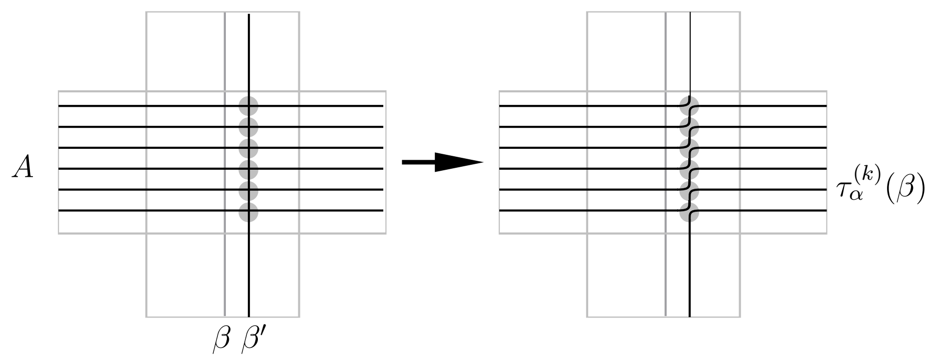

Figure 3.2: A standard twist.

By taking $n = k \cdot i(\alpha, \beta)$ with $k > 0$ in the above construction, we get the k-fold twist, or $\tau_\alpha^{(k)}(\beta)$. Now we introduce some more terminology, followed by some remarks about twists which we will need later.

Any union of curves meeting essentially is easily seen to be an embedding of a graph whose vertices are the intersection points. In the following, we do not make a distinction between graphs and unions of curves, and consider the graph structure implicit.

The **complementary regions** or **regions** of a graph G in S are the components of S/G. The **sides** of a region R are the edges of G contained in ∂R, where we count an edge e twice if there is an open neighborhood of some point on e contained in the closure of R. When G is the union of curves, we can refer to a side of a region R as an x-side, where x is the curve in G containing the side.

Now we analyze the regions of $\tau_\alpha^{(k)}(\beta) \cup \beta$ where $k > 0$. Let A and β' be as in the definition above. Let B be a closed annulus containing β and β'.

Each component of $S/(A \cup B)$ is contained in exactly one region of $\alpha \cup \beta$, and conversely each region of $\alpha \cup \beta$ contains exactly one component of $S/(A \cup B)$.

Let R be a region of $\alpha \cup \beta$ and C the component of $S/(A \cup B)$ contained in it. Then $R = (\bigcup_s D_s) \cup (\bigcup_t E_t) \cup C$, where s runs over the α-sides of R, t runs over the β-sides of R, and $D_s \subseteq R \cap A/B$ and $E_t \subseteq R \cap B$ are half-open rectangular components; see Figure 3.3.

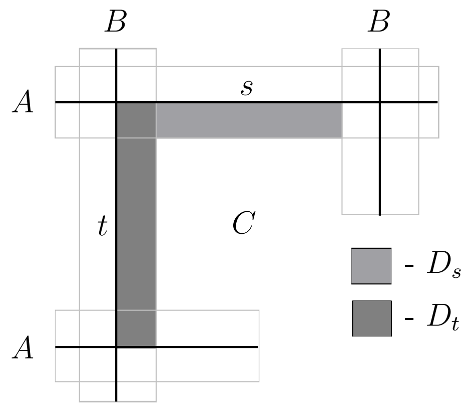

Figure 3.3: Region of $\alpha \cup \beta$

C is also contained in exactly one region, R', of $\tau_\alpha^{(k)}(\beta) \cup \beta$. We can similarly write $R' = (\bigcup_{s'} D'_{s'}) \cup (\bigcup_{t'} E'_{t'}) \cup C$, where s' runs over the $\tau_\alpha^{(k)}(\beta)$-sides of R', t' runs over the β-sides of R', and $D'_{s'} \subseteq R' \cap A/B$ and $E'_{t'} \subseteq R' \cap B$ are half-open rectangular components; see Figure 3.4.

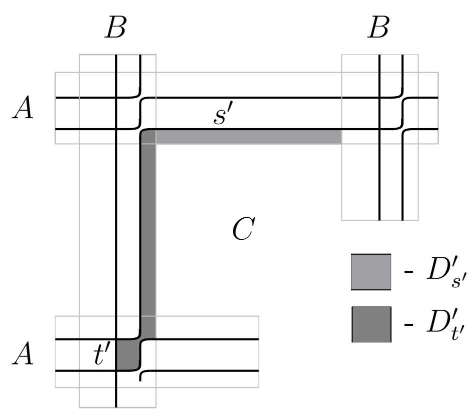

Figure 3.4: Region of $\tau_\alpha^{(k)}(\beta) \cup \beta$

So R' has the same number of sides as R. Also notice that the regions of $\tau_\alpha^{(k)}(\beta) \cup \beta$ which *do not* contain any component of $S/(A \cup B)$ are contained in $A \cup B$ and are rectangular (have four sides). In particular, if $\alpha \cup \beta$ has no bigon complementary regions, neither does $\tau_\alpha^{(k)}(\beta) \cup \beta$.

We will also need the fact that, assuming $i(\alpha, \beta) \geq 2$, every arc of $\beta/\tau_\alpha^{(k)}(\beta)$ has a parallel copy. In other words, if $i(\alpha, \beta) \geq 2$ then every β-edge of $\tau_\alpha^{(k)}(\beta) \cup \beta$ is adjacent to a rectangular region of $\tau_\alpha^{(k)}(\beta) \cup \beta$ on at least one side. This fact follows from a study of Figure 3.4.

Now we can return to the construction. From this point on, we assume all embedded graphs / unions of curves fill S, i.e. all complementary regions are open disks.

Definition 3.1.2. *Suppose α, β, and γ are simple closed curves, a a closed arc. If $a \subseteq \alpha \subseteq (\gamma \cup a)$, $a \cap \beta = \emptyset$, and α, γ meet β essentially, we write $\alpha \prec_\beta \gamma$.*

The symbol $\prec$ can be read as "almost contained in", and the usefulness of the idea emerges when $\alpha \prec_\beta \gamma$ and there is a curve γ' so that $\gamma \cap \gamma' = \emptyset$. If this is the case, then γ' can intersect α in at most one arc of α/β, namely the arc containing a. So in a sense, "γ' is almost disjoint from α". The next lemma gives our first application of this idea.

For the following lemma, fix α and β so that $\alpha \cup \beta$ fills S, and define $\alpha_0 = \alpha$, $\alpha_{n+1} = \tau_{\alpha_n}^{(2)}(\beta), n \geq 0$.

Lemma 3.1.3. *If $n > 1$, $\alpha_n \prec_\beta \gamma'$ and $\gamma \cap \gamma' = \emptyset$, then $\alpha_{n-1} \prec_\beta \gamma$.*

Proof. Let A, B, and β' be as in the previous discussion for the twist of β about α_{n-1}. We may assume that γ meets ∂A and ∂B efficiently. Since

$\alpha_{n-1} = \tau^{(2)}_{\alpha_{n-2}}(\beta)$, every arc of β/α_{n-1} has a parallel copy. Clearly, then, every arc of β/A has a parallel copy as well. Each arc of β/A is also parallel to a unique arc of α_n (one contained in β'), so in fact, every arc of α_n/A has a parallel copy. The is best seen by studying Figure 3.4.

By hypothesis, γ can intersect α_n in at most one arc of α_n/β and therefore in at most one arc of α_n/A. From the previous paragraph and the fact that $\alpha_n \cup A$ fills S, γ must intersect ∂A.

Now we analyze an arc $c \subseteq \gamma \cap A$. Orient the core of A and number the arcs of $(\partial B \cap A)/\alpha_n$ to increase from 0 on the right component of ∂A to $2i(\alpha_{n-1}, \beta)$ on the left component of ∂A, as in Figure 3.5 (a).

For the purpose of this argument, we may assume there are no intersections of γ with α_n in $A \cap B$. So in each component of $A \cap B$, c must increase in "height" by 1 from left to right, as in Figure 3.5.

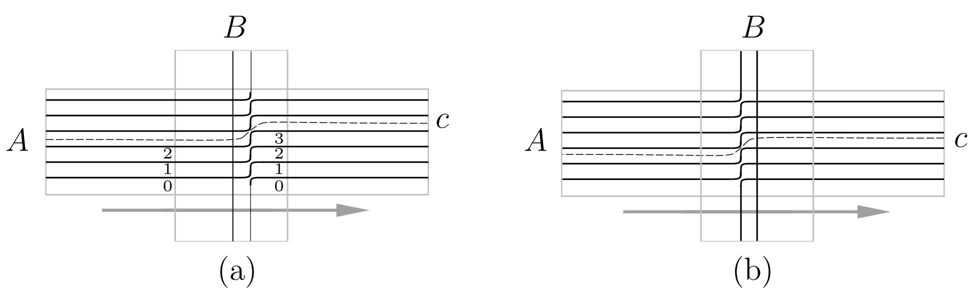

Figure 3.5: Forcing of c in $A \cap B$.

In each component D of A/B, either c remains at the same height (Figure 3.6 (a)) or c meets the one arc of α_n/β which it can according to hypotheses in D and increases its height by 1 (Figure 3.6 (b)).

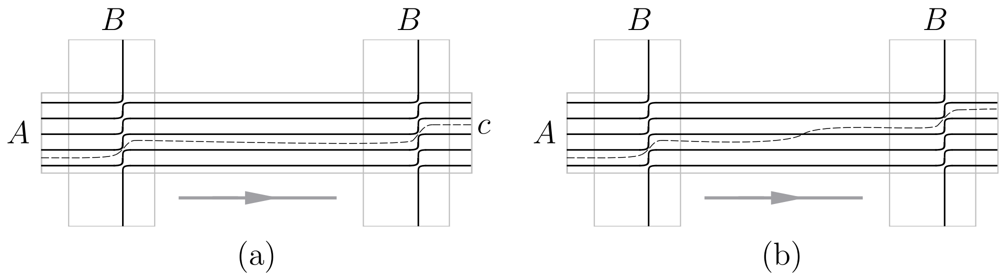

Figure 3.6: c forced in A/B

So in order for c to wind its way from one component at height 0 to the other at height $2i(\alpha_{n-1}, \beta)$ it must meet some component, D_0, of A/B twice. Therefore, there is a subarc $c' \subseteq c \subseteq \gamma$ and an arc $a \subseteq D_0$ such that we can take α_{n-1} to be $a \cup c'$ (see Figure 3.7), and we have $\alpha_{n-1} \prec_\beta \gamma$. □

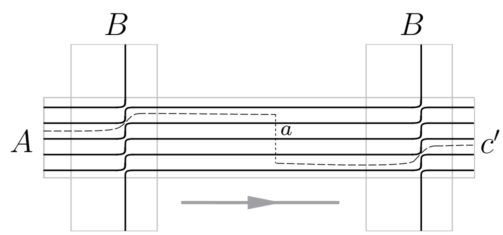

Figure 3.7: Construction of $\alpha_{n-1} = a \cup c'$.

Proposition 3.1.4. *[6, 10] The curve complex has infinite diameter*

Proof. Let α_N and β be as above. We show that $d(\beta, \alpha_N) > N$. Suppose, for a contradiction, that there is a path $\beta = \gamma_0, \cdots, \gamma_N = \alpha_N$.

We claim that $\alpha_n \prec_\beta \gamma_n$ for $n \geq 1$. This follows by induction from Lemma 3.1.3, noting that it is obviously true for $n = N$. The contradiction now comes from the fact that $\alpha_1 \prec_\beta \gamma_1$ implies γ_1 must intersect β, whereas we assumed $\beta = \gamma_0, \gamma_1, \ldots$ was a path in $\mathcal{C}(S)$. □

Now we use a similar technique to prove the following:

Proposition 3.1.5. *Given a pair of curves α and β with $n = d(\alpha, \beta)$, there is a sequence $\alpha = \gamma_1, \dots, \gamma_n = \beta$ in $\mathcal{C}(S)$ such that $i(\beta, \gamma_i)$ is strictly decreasing in i and $i(\gamma_i, \gamma_{i+1}) \leq 1$ for $i = 1, \dots, n-1$.*

First we need some definitions.

Definition 3.1.6. *Let G be a graph embedded in S. A curve which meets each complementary region of G at most once is called a G-**cycle** (e.g. a cycle in the dual graph of G).*

*For an arbitrary curve, γ, α is a G-**cycle of** γ if α is a G-cycle and there is an arc, a, disjoint from G such that $a \subseteq \alpha \subseteq (\gamma \cup a)$. We write $\alpha \ll_G \gamma$ in this case.*

Remark. In other applications, we might require that α and γ meet G "essentially" somehow. In the following, however, this type of requirement would be redundant.

Also note that for any curve γ there is a G-cycle of γ. Either γ itself is a G-cycle, or we take a minimal subarc of γ which meets some complementary region of G more than once and connect its ends with an arc contained in such a region.

Proof of Proposition 3.1.5. Suppose $\alpha = \gamma_0, \dots, \gamma_n = \beta$ is a geodesic. Put $\gamma'_0 = \alpha$, and for $i = 1, \dots, n$ take γ'_i to be an essential $(\beta \cup \gamma'_{i-1})$-cycle of γ_i. Notice that $i(\gamma'_i, \beta) < i(\gamma'_{i-1}, \beta)$ for $1 \leq i \leq n$.

Since $\gamma'_{i-1} \prec_\beta \gamma_{i-1}$ and $\gamma_i \cap \gamma_{i-1} = \emptyset$, γ_i can intersect γ'_{i-1} in at most one arc of γ'_{i-1} cut along β (the arc containing a from the definition of $\prec$). The *cycle* γ'_i, of γ_i, therefore will meet γ'_{i-1} in that one arc in at most one point, since it meets the adjacent regions at most once. So $i(\gamma'_{i-1}, \gamma'_i) \leq 1$, concluding the proof of Proposition 3.1.5. $\square$

The number of $(\beta \cup \gamma'_{i-1})$-cycles is finite, so the possibilities for γ'_i given γ'_{i-1} are *finite*. Therefore, this process can be made into an algorithm. We can compute the distance between α and β up to a factor of two by choosing the shortest length path of cycles as in the proposition, having decreasing intersections with β. If we aim primarily to approximate distance, however, we have a better method.

Fix a graph G embedded in S. Let $c(G)$ denote the set of G-cycles. We will now show that between any two points of $c(G)$ there is a $(2, 2)$ quasi-geodesic contained in $c(G)$. Since $c(G)$ is finite, we get an algorithm for approximating the distance between points of $c(G)$.

The following proposition is the fundamental idea, allowing for a "projection" of $\mathcal{C}(S)$ onto $c(G)$.

Proposition 3.1.7. *If* $\alpha \ll_G \gamma$, $\alpha' \ll_G \gamma'$, *and* $\gamma \cap \gamma' = \emptyset$ *then* $i(\alpha, \alpha') \leq 2$.

Proof. Let a be the arc from the definition of $\ll$ corresponding to α and γ, similarly define a' to be the arc corresponding to α' and γ'. The only place α'/a' can intersect α is in a. Since α' is a cycle and a is contained in one region, α'/a' can meet α in at most one point. Similarly α/a can meet α' in at most one point, and the proposition follows. $\square$

If $\gamma_0, \dots, \gamma_n$ is a geodesic in $\mathcal{C}(S)$ such that $\alpha_0 = \gamma_0$ and $\alpha_n = \gamma_n$ are G-cycles, we can choose arbitrary G-cycles, α_i, of the remaining γ_i. Then for any i, j such that $0 \leq i, j \leq n$, the previous proposition implies $d(\alpha_i, \alpha_j) \leq 2|i-j|$. If $d(\alpha_i, \alpha_j) < |i-j|$ we replace $\alpha_{i+1}, \dots, \alpha_{j-1}$ with cycles of a geodesic from α_i to α_j. Repeating this process sucessively, we eventually get *a $(2,2)$ quasi-geodesic of G-cycles.*

Given curves α and β, we approximate their distance by choosing $G = \alpha \cup \beta$. In that case, there are G-cycles disjoint from α and β, so α and β are close to $c(G)$ in $\mathcal{C}(S)$. If we consider elements of $c(G)$ adjacent when they intersect in two or fewer points, then the resulting graph structure on $c(G)$ approximates a portion of $\mathcal{C}(S)$ which includes α and β. Since $c(G)$ is finite, we can compute minimum length paths. Notice also, for this choice of G, if $x \in c(G)$ then $i(x, \alpha), i(x, \beta) < i(\alpha, \beta)$.

3.2 Second method

We fix a set of curves and consider intersections with a fixed "reference" arc. The set up is *not* isotopy invariant; since an arc is contractible, any reference arc can be isotoped to be disjoint from any set of curves. The condition of meeting a reference arc, A, *essentially*, as in the following definition, along with the *number* of intersections with A are, however, invariant under isotopies supported on the complement of ∂A. Later we will deduce isotopy invariant results which apply to the curve complex proper.

To get the most general result, we use a fairly general object to calculate intersections:

Definition 3.2.1. *Let X be a collection of curves embedded in S. Let A be an embedding of a closed arc in S meeting each $x \in X$ transversely. We say that A meets X* **essentially** *if for each $x \in X$ there are no subarcs $a \subseteq A, b \subseteq x$ so that $a \cup b$ bounds a disk in S. In other words, X and A meet essentially in $S/\partial A$.*

Notice that any subarc of an arc which meets X essentially also meets X essentially. Notice also, if y is a curve which meets each $x \in X$ essentially, then any subarc of y meets X essentially.

For the remainder of the section, $[x_i] \in \mathcal{C}(S)$ so that $[x_0] \perp \cdots \perp [x_n]$ and the $\{x_i\}$ meet essentially. A is an arc in S which meets $\{x_i\}$ essentially. Fix i_0 so that $0 < i_0 < n$.

Let I be a subarc of A. We inductively label certain points of intersection of $\bigcup_{0\leq j\leq n} x_j$ with I as "**good with respect to** I" or "**good**" when I is understood. Any point in $x_{i_0} \cap I$ is good. If $0 \leq j < i_0$, $p \in (x_j \cap I)$, and $q, r \in (x_{j+1} \cap I)$ such that p is between q and r on I, and q and r are good then p is good. Similarly, if $i_0 < j \leq n$, $p \in (x_j \cap I)$, and $q, r \in (x_{j-1} \cap I)$ such that p is between q and r on I, and q and r are good then p is good. We say that $x_{j_0}, \ldots, x_{j_1}$ **meets** I **well** if $j_0 \leq i_0 \leq j_1$ and there are good points in $x_{j_0} \cap I$ and $x_{j_1} \cap I$.

Define $B = 6(6g-2)+2$, and let $\mathcal{P}(k)$ represent the following assertion for non-negative k:

$\mathcal{P}(k)$: $|x_{i_0} \cap A| \geq B^k$ and for each subarc $I \subseteq A$ such that $|x_{i_0} \cap I| \geq B^k$ there is a subsequence $x_{j_0}, \ldots, x_{j_1}$ of length k which meets I well.

Theorem 3.2.2. *Suppose* $|x_0 \cap A| = |x_n \cap A| = 0$. *Choose* $k_0 \geq 0$ *minimal so that* $\mathcal{P}(k_0)$ *is false. Then either:*

(1) $|x_{i_0} \cap A| < B^{k_0}$, *or*

(2) *there are subarcs* $a \subseteq A, b \subseteq x_{i_0}$ *such that* $y = a \cup b$ *is an essential simple closed curve and* $[x_{j_0}] \perp [y] \perp [x_{j_1}]$ *where* $0 \leq j_0 < i_0 < j_1 \leq n$ *and* $j_1 - j_0 = k_0 + 1$. *In particular, after an isotopy supported on* $S/\partial A$, $|y \cap A| < |x_{i_0} \cap A|$.

Remark. The statement of the theorem is in terms of the "least criminal", k_0, which either bounds $|x_{i_0} \cap A|$ or measures how much we can reduce the path $x_0, \ldots, x_n$. All of the corollaries hide k_0, so we mention it here as a point of interest. Assuming $|x_{i_0} \cap A|$ is large, if $k_0 = 1$, we can replace x_{i_0} with a "better" curve - one which has fewer intersections with the reference arc. If $k_0 > 2$, then $x_0, \ldots, x_n$ can be reduced to a path shorter by $k_0 - 1$. The larger k is when $\mathcal{P}(k)$ fails, the further from being geodesic the given path is. We might wonder if, under the right hypotheses, the failure of $\mathcal{P}$ could be used to show that a given path is close to being geodesic.

Proof of theorem. Assume $|x_{i_0} \cap A| \geq B^{k_0}$. If $|x_{i_0} \cap A| > 0$ then $\mathcal{P}(0)$ is true trivially. So $k_0 \geq 1$ and $\mathcal{P}(k_0 - 1)$ is true. $\mathcal{P}(k_0)$ false implies there is a subarc $I \subseteq A$ such that there is no k_0 length subsequence which meets I well. By $\mathcal{P}(k_0 - 1)$, however, there is a $k_0 - 1$ length subsequence, $x_{j_0}, \ldots, x_{j_1}$ which meets I well. We are assuming $x_0 \cap A = x_n \cap A = \emptyset$, so $j_0 > 0$ and $j_1 < n$. Also note that there are no good points in $x_{j_0 - 1} \cap I$ or $x_{j_1+1} \cap I$. For each $J \subseteq I$ with $|x_{i_0} \cap J| \geq B^{k_0 - 1}$ there is some $k_0 - 1$ length sequence $x_{j'_0}, \ldots, x_{j'_1}$ which meets J well, and from the previous statements we must have $j_0 - 1 < j'_0 \leq i_0 \leq j'_1 < j_1 + 1$, so in fact $j'_0 = j_0$ and $j'_1 = j_1$.

Let $\mathcal{J}$ be a disjoint collection of B subarcs of I so that for each $J \in \mathcal{J}$, $|x_{i_0} \cap J| \geq B^{k_0 - 1}$. Then $x_{j_0}, \ldots, x_{j_1}$ meets each $J \in \mathcal{J}$ well. Let J' and J'' be the outermost arcs of $\mathcal{J}$, $\mathcal{J}' = \mathcal{J} - \{J', J''\}$, and $I' = I/(J' \cup J'')$. Then since each of x_{j_0} and x_{j_1} meet both J' and J'', any point of intersection of $x_{j_0 - 1}$ or x_{j_1+1} with I' is good. There are no good points though, so $x_{j_0-1} \cap I' = x_{j_1+1} \cap I' = \emptyset$.

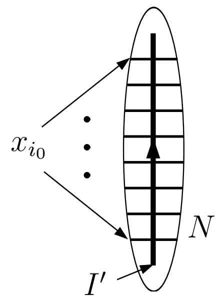

Figure 3.8: Choice of N.

Choose a neighborhood N of I' so that $N \cap (\bigcup_{0<i<n} x_i)$ is a collection of disjoint arcs (see Figure 3.8). Fix an orientation of I'. For each $J \in \mathcal{I}'$ choose a point p_J of $x_{i_0} \cap J$. Define a_J to be the arc of x_{i_0}/N which extends into N on the right side (say) to meet I' at p_J. By an Euler characteristic argument, some six of the $\{a_J\}$ are parallel in S/N. At least half of these parallel arcs represent *distinct* arcs, i.e. have disjoint endpoints. Label $J_1, J_2, J_3 \in \mathcal{I}'$ so that J_2 is between J_1 and J_3, and $a_{J_1}, a_{J_2}, a_{J_3}$ are distinct parallel arcs.

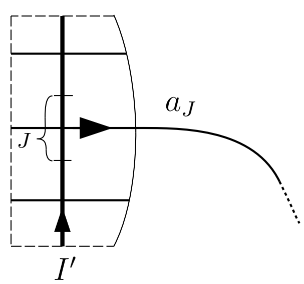

Figure 3.9: Choice of a_J

Let R be the rectangle with "horizontal" sides a_{J_1}, a_{J_3}, and "vertical" sides two arcs of ∂N as in Figure 3.10.

We claim now that we can extend R into N to get a rectangle R' whose vertical sides are subarcs of I' and horizontal sides are arcs of x_{i_0}/I'. In other words, we claim that each vertical side of R bounds a rectangle with a subarc of I' and subarcs of $x_{i_0} \cap N$ (shown as the darker shaded regions in Figure 3.10). The only obstructions would contradict that A meets x_{i_0} essentially.

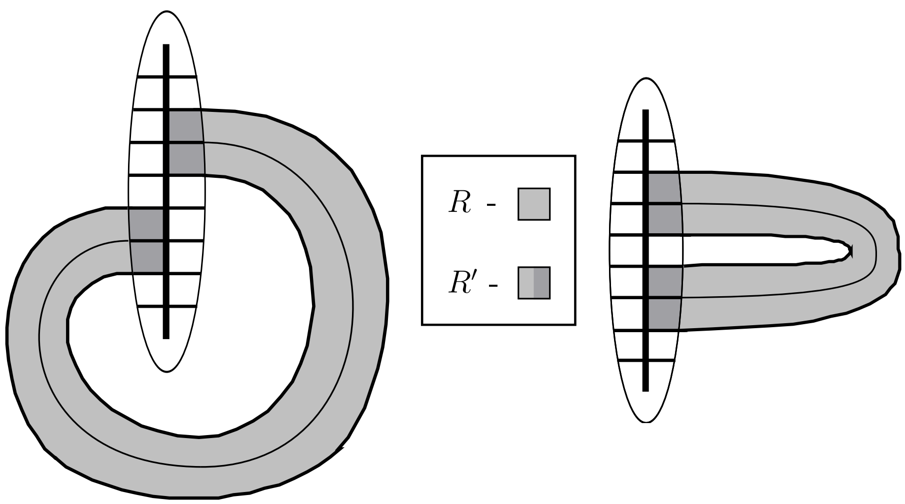

Figure 3.10: Constructing R and R'

Since x_{i_0} has no self intersections and meets A essentially, we must have that all of the points of $x_{i_0} \cap J_2$ are the (left) endpoints of horizontal arcs in $x_{i_0} \cap R'$. Now we inductively prove that for good $p \in x_j \cap J_2$, the arc of $x_j \cap R'$ which meets p is horizontal in R'. If $j < i_0$ and $p \in x_j \cap J_2$ is good, there are good points $q, r \in x_{j+1} \cap J_2$ which correspond to horizontal arcs in R'. Since $x_j \cap x_{j+1} = \emptyset$ and x_j meets A essentially, the arc from p in R' is "sandwiched" between the arcs from q and r, and is therefore horizontal. Similarly for $j > i_0$.

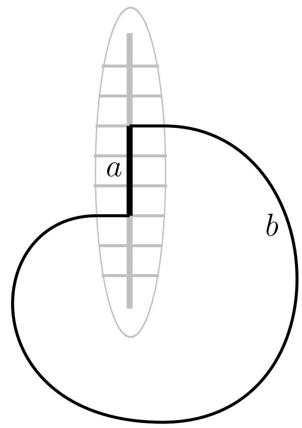

Figure 3.11: Construction of y

So for each $j_0 \leq j \leq j_1$, there is a horizontal arc of $x_j \cap R'$. In particular this means there are no vertical arcs $x_{j_0-1} \cap R'$ or $x_{j_1+1} \cap R'$. Now glue some horizontal arc b in R' to the arc its endpoints bound, $a \subseteq I'$, to get a curve y (see Figure 3.11). y is essential because x_{i_0} meets A essentially. Since $a \subseteq I'$, $x_{j_0-1} \cap a = x_{j_1+1} \cap a = \emptyset$. Along with the fact that there are no vertical arcs of x_{j_0-1} or x_{j_1+1} in R', we have $[x_{j_0-1}] \perp [y] \perp [x_{j_1+1}]$, and conclusion (2) follows. $\square$

3.2.1 Corollaries

All of the corollaries involve applying the theorem to a set of disjoint reference arcs simultaneously. The first corollary employs this idea in order to provide a version of the main theorem for more general arcs.

Corollary 3.2.3. *(Allowing for $|x_0 \cap A|, |x_n \cap A| \geq 0$) Suppose $x_0, \ldots, x_n$ forms a path in the curve complex which meets A essentially and $0 < i_0 < n$. Then*

either (1) $|x_{i_0} \cap A| \leq B^n(|x_0 \cap A| + |x_n \cap A|)$,

or (2) *there are subarcs $a \subseteq A, b \subseteq x_{i_0}$ such that $y = a \cup b$ is an essential simple closed curve and $[x_{j_0}] \perp [y] \perp [x_{j_1}]$ where $0 \leq j_0 < i_0 < j_1 \leq n$. In particular, after an isotopy supported on $S/\partial A$, $|y \cap A| < |x_{i_0} \cap A|$.*

Proof. Divide A along its intersections with $x_0 \cup x_n$ and apply the main theorem on each piece. □

The remaining corollaries apply the main theorem to find "good" geodesics with respect to different collections of reference arcs. In each case, we inductively "reduce" a path using (2) of the main theorem until (1) holds for every vertex. Since the "reduced curve", y from the theorem, is the union of a subarc of a reference arc with x_{i_0}, we can reduce our path for one reference arc without increasing intersections with the others.

The next corollary gives this section's best attempt at bounding the intersection numbers of the vertices along a geodesic with its endpoints.

Corollary 3.2.4. *Given $\alpha, \beta \in \mathcal{C}(S)$, there is a geodesic $\alpha = x_0, \ldots, x_n = \beta$ such that for each k,*

$$i(\alpha, x_k), i(x_k, \beta) \quad \leq \quad B^n i(\alpha, \beta)$$

$$i(\alpha, x_k) \leq B^k i(\alpha, x_{k+1}) \quad \textit{and} \quad i(x_k, \beta) \leq B^{n-k} i(x_{k-1}, \beta)$$

Proof. Fix a geodesic $\alpha = x_0, \ldots, x_n = \beta$. Applying the theorem iteratively to $x_0, \ldots, x_n$, we can assume the first part of the corollary holds. Then for each k, apply the theorem to both $x_0, \ldots, x_{k+1}$ and $x_{k-1}, \ldots, x_n$. $\square$

We can also apply Corollary 3.2.4 to get bounds in a fixed frame of reference, namely we can bound Dehn-Thurston parameters. [8]

Corollary 3.2.5. *Fix a Dehn-Thurston coordinate system. Let x and y be points of $\mathcal{C}(S)$. Then there is a geodesic between x and y so that all interior vertices, z, have Dehn-Thurston (twist) coordinates bounded by a multiple of the sum of the corresponding Dehn-Thurston (twist) coordinates of x and y. In particular, for each (twist) coordinate $c : \mathcal{C}(S) \to \mathbb{Z}$, $|c(z)| \leq 2i(x,y)^{ln(B)/ln(2)}(|c(x)| + |c(y)|)$.*

Proof. We need only show that when in normal form a curve meets the appropriate reference arcs essentially, then apply the first corollary. The reader is referred to [8]. $\square$

Given a large train track, an arc crossing a branch can act as a reference arc for Corollary 3.2.3 and we can bound the measures of points along paths carried by the track. We just need to verify that the y curves constructed in Corollary 3.2.3 are carried by the track. [13]

Now consider the bounds enforced by Corollary 3.2.4 and Corollary 3.2.5. Assuming that $\alpha \cup \beta$ fills in the first case, and that we bound all coordinates in the second, there are a finite number of bounded curves. So we can compute minimum length paths, and we get the following corollary.

Corollary 3.2.6. *There is an algorithm to compute the distance between curves in* $\mathcal{C}(S)$.

We do not mention this in the belief that anyone will ever implement it. The novelty is that finding the exact distance between two curves in the curve complex should be so awkward.

3.3 Conclusion

The first section gives a method for "projecting" the curve complex onto certain finite subsets associated with embedded graphs. Does any kind of contraction property as in [10] hold? Given that $\alpha \ll_G \gamma$, we would like a way to estimate $d(\alpha, \gamma)$ to approach questions like this. Also, can we further restrict the set of cycles which would occur as cycles of curves along a quasi-geodesic? Is there, furthermore, a parameter *selecting* the cycles along a quasi-geodesic?

In the second section, we give a method for computing the exact distance between two points of $\mathcal{C}(S)$ which is much more computationally expensive than the method for obtaining an estimate in the first section. Is this a limit of the method or a property of the curve complex? Also, the remark following Theorem 3.2.2 on $\mathcal{P}(k_0)$ gave a guess at what a converse might look like. Does this lead to a local property of a path in $\mathcal{C}(S)$ that guarantees it is (quasi) geodesic?

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Vita

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